Ch 2: Differentiation

2.1 The geometry of real valued functions

A function f that takes n inputs and gives m outputs

is called **vector valued** if $m > 1$

and **scalar valued** if $m = 1$

We write $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$

(A: domain of f; $\mathbb{R}^m$: range)

and $\vec{x} \mapsto f(\vec{x})$

example: the function $f: \mathbb{R}^3 \to \mathbb{R}$

given by $(x,y,z) \mapsto x^2+y^2+z^2$

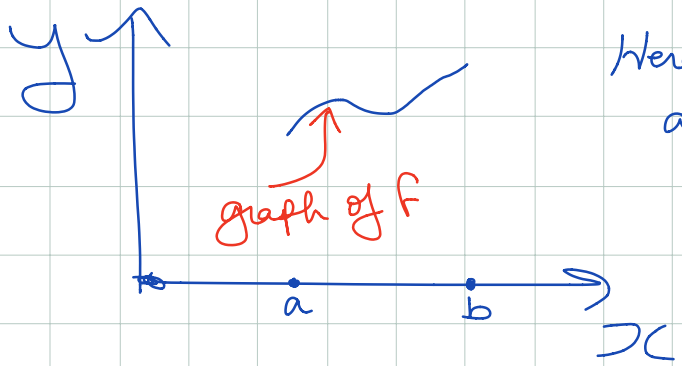
is scalar valued, whereas

$f: \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x^2+y^2+z^2, x+y+z)$

is vector valued

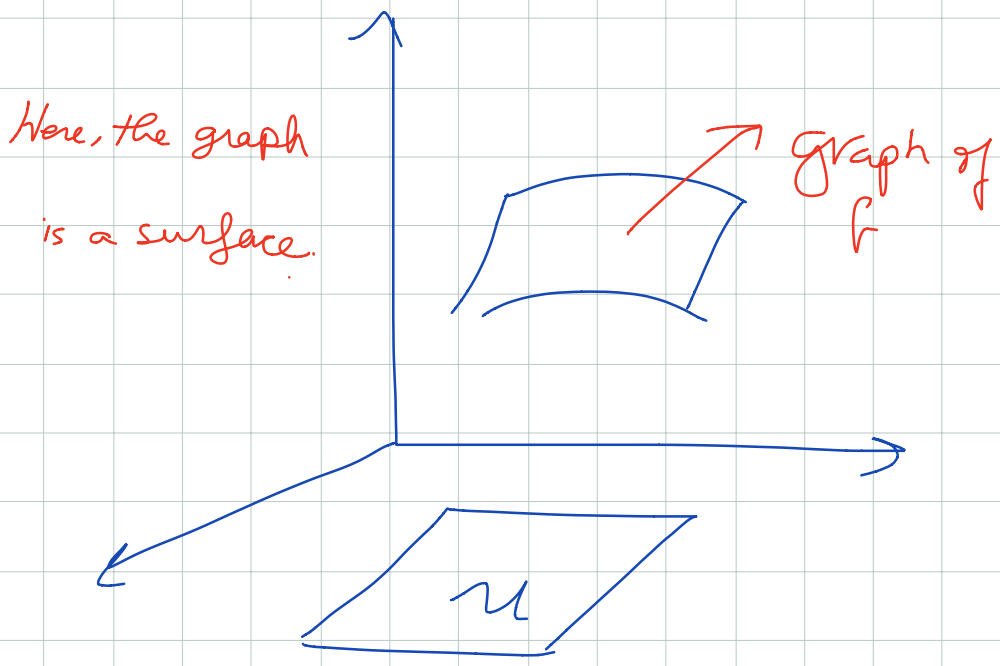
We can associate to a function $f: U \subset \mathbb{R}^n \to \mathbb{R}$ a graph

e.g.: $f: (a,b) \subset \mathbb{R} \to \mathbb{R}$



Here, the graph is a curve

e.g. $f: U \subset \mathbb{R}^2 \to \mathbb{R}$



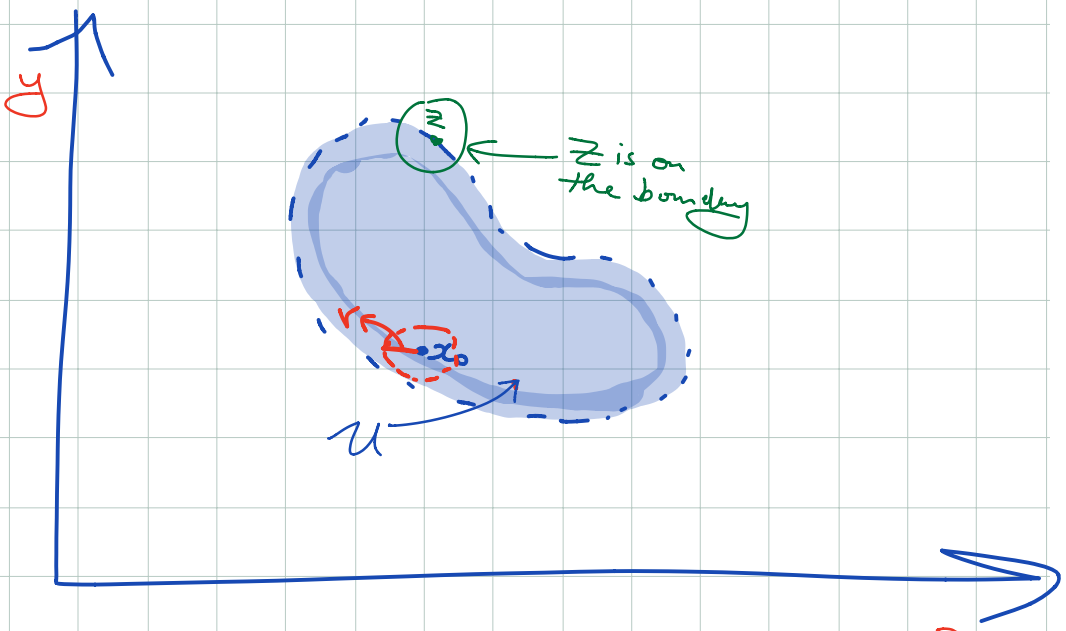
2.2 Limits & Continuity

We'll need a couple of definitions before we talk about limits.

Open set: Let U be a subset of $\mathbb{R}^n$ (written $U \subset \mathbb{R}^n$). We say that U is an open set if for every x_0 in U there is some number $r > 0$ such that every point with $\|x - x_0\| < r$ is within U.

e.g.: any interval $(a,b) \subset \mathbb{R}$ is open.

e.g.:



Intuitively: U is open when the boundary points of U are not in U.

(a point $z \in U$ is on the boundary of U if every neighborhood of z contains at least one pt. in U & one pt. not in U.)

Limits

Remember that in "standard" one dimensional calculus, we used limits to study continuity, define derivatives, improper integrals, ...

We would like to generalize this notion to functions of several variables.

Definition: Let $A \subset \mathbb{R}^n$ be an open set and let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$. Let $\vec{x}_0$ be in A or be on the boundary of A.

We write $\lim_{x \to x_0} f(\vec{x}) = \vec{b}$

when given any neighborhood N of $\vec{b}$ (i.e. an open set containing b) f is eventually in N as $\vec{x}$ approaches $\vec{x}_0$.

If f does not approach any vector as $x \to x_0$, we say the limit does not exist.

Properties of limits

- If $\lim_{x \to x_0} f(\vec{x}) = \vec{b}_1$ & $\lim_{x \to x_0} f(\vec{x}) = \vec{b}_2$ then $\vec{b}_1 = \vec{b}_2$
- If $\lim_{x \to x_0} f(\vec{x}) = \vec{b}_1$ & $\lim_{x \to x_0} g(\vec{x}) = b_2$

then

- $\lim_{x \to x_0} c f(\vec{x}) = c\vec{b}$
- $\lim_{x \to x_0} \left(f(\vec{x}) + g(\vec{x}) \right) = \vec{b}_1 + \vec{b}_2$
- when $m = 1$ (i.e. b_1 & b_2 are scalars)
  $\lim_{x \to x_0} f(\vec{x}) g(\vec{x}) = b_1 b_2$
- when $f(\vec{x}) = \left(f_1(x), f_2(x), \ldots f_m(x) \right)$
  then $\lim_{x \to x_0} f(\vec{x}) = \vec{b} = (b_1, b_2, \ldots b_m)$

  if and only if $\lim_{x \to x_0} f_i(x) = b_i$

  for $i = 1, 2, \ldots, m$.

Examples: Let $f: \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 + y^2$

Compute $\lim_{(x,y) \to (0,1)} f(x, y)$

Solution: $\lim_{(x,y)\to(0,1)} f(x,y) = \lim_{(x,y)\to(0,1)} x^2+y^2$

$$= \lim_{(x,y)\to(0,1)} x^2 + \lim_{(x,y)\to(0,1)} y^2$$

$$= \left(\lim_{(x,y)\to(0,1)} x\right)^2 + \left(\lim_{(x,y)\to(0,1)} y\right)^2$$

$$= 0^2 + 1^2 = 1$$

Example: Use polar coordinates to find the limit (if it exists)

$$\lim_{(x,y)\to(0,0)} \frac{7x^3}{x^2+y^2}$$

Solution: $\lim_{(x,y)\to(0,0)} \frac{7x^3}{x^2+y^2} = \lim_{r\to 0} \frac{7r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$

$$= \lim_{r\to 0} 7r\cos^3\theta = 0.$$

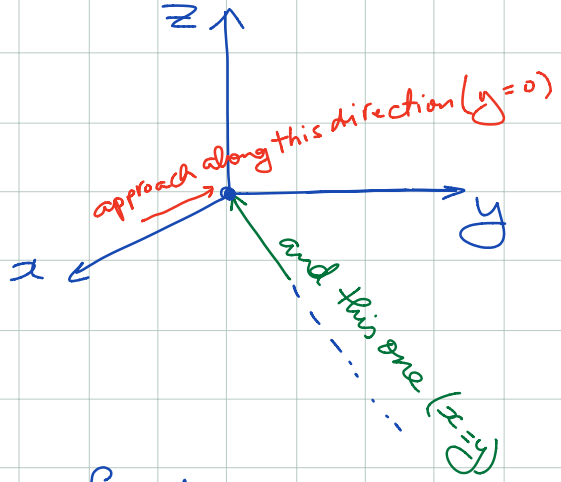
Example: Find the limit or show it doesn't exist

$$\lim_{(x,y)\to(0,0)} \frac{7x^2}{x^2+y^2}$$

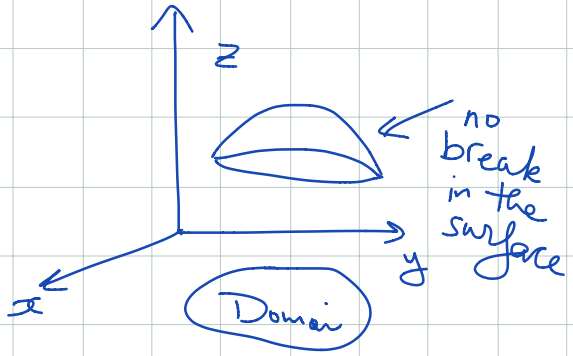
Soln: To show the limit doesn't exist, we can approach $(0,0)$ from 2 different directions. First, fixing $y=0$ and letting $x\to 0$ we get $\lim_{(x,y)\to(0,0)} \frac{7x^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{7x^2}{x^2} = 7$

Second, fixing $x=y$, we get $\lim_{(x,y)\to(0,0)} \frac{7x^2}{x^2+y^2} = \frac{7}{2}$

$\Rightarrow$ does not exist



Continuous functions:



Defn: Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function and let $x_0 \in A$. We say that f is continuous at x_0 if & only if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

If f is continuous at every point in A, we say that f is continuous.

Example: $f(x) = 2x^2 + 3x + 5$ is continuous (why?)

Example: $f(x,y) = xy$ is continuous

because $\lim_{(x,y)\to(x_0,y_0)} xy = \left(\lim_{(x,y)\to(x_0,y_0)} x\right)\left(\lim_{(x,y)\to(x_0,y_0)} y\right)$

$= x_0 y_0 = f(x_0, y_0)$

for all points (x_0, y_0)

Example: $f(x,y) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is <u>not</u> cont.

(why? Think about $\lim_{(x,y)\to(0,0)} f(x,y)$)

Properties: Suppose f & g are continuous at x_0:

- cf is also continuous at x_0 (c is a real number)
- $f+g$ is also continuous at x_0
- when f & g are functions from $\mathbb{R}^n$ to $\mathbb{R}$, fg is continuous at x_0
- Let $f(x) = (f_1(x), f_2(x), \cdots f_m(x))$; then f is cont. if and only if the f_i's are all cont.

• Compositions of continuous functions are continuous:

If g is continuous at x_0 & f is continuous at $y_0 = g(x_0)$ the $f \circ g$ is continuous at x_0

(Recall that $(f \circ g)(x) = f(g(x))$.

2.3 Differentiation:

Partial derivatives

Recall from single variable calculus that we defined for $f: \mathbb{R} \to \mathbb{R}$ the derivative as

$$\frac{df}{dx}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ = rate of change of f as x changes

(when the limit exists)

When we have a function of several variables, i.e. $f: \mathbb{R}^n \to \mathbb{R}$ (for example $f(x,y) = x^2 + y^2$) we can define the rate of change of f in each of the n direction (in our example it would be x & y)

Let $U \subset \mathbb{R}^n$ be open and suppose $f: U \subset \mathbb{R}^n \to \mathbb{R}$

then the partial derivatives of f at the point $(x_1, x_2, \ldots, x_n)$ are defined by

$$\frac{\partial f}{\partial x_1}(x_1, x_2, \ldots x_n) = \lim_{h \to 0} \frac{f(x_1+h, x_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)}{h}$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2, \ldots x_n) = \lim_{h \to 0} \frac{f(x_1, x_2+h, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)}{h}$$

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \ldots x_n) = \lim_{h \to 0} \frac{f(x_1, x_2 \ldots, x_j+h \ldots, x_n) - f(x_1, x_2, \ldots, x_j, \ldots, x_n)}{h}$$

$$= \lim_{h \to 0} \frac{f(\vec{x} + h\vec{e}_j) - f(\vec{x})}{h}$$

here $e_j = (0, 0, \ldots, 1, 0, \ldots)$ (1 in the jth position)

Example: Let $f(x,y) = x^2 + xy^3$

$$\frac{\partial f}{\partial x} = \frac{\partial (x^2 + xy^3)}{\partial x} = 2x + y^3$$

(think of y as just a constant)

$$\frac{\partial f}{\partial y} = \frac{\partial (x^2 + xy^3)}{\partial y} = 3xy^2$$

think of x as just a const.

Example: Let $z = \ln(x^5 + y^4)$ Find $\left.\frac{\partial z}{\partial x}\right|_{(1,1)}$ & $\left.\frac{\partial z}{\partial y}\right|_{(1,1)}$

Soln:

$$\frac{\partial z}{\partial x} = \frac{1}{x^5+y^4} \frac{\partial}{\partial x}(x^5+y^4) = \frac{5x^4}{x^5+y^4}$$

by Chain rule

$$\frac{\partial z}{\partial y} = \frac{1}{x^5+y^4} \frac{\partial}{\partial y}(x^5+y^4) = \frac{4y^3}{x^5+y^4}$$

(All standard rules of differentiation apply)